

# COMBINATORIAL PROPERTIES OF VIRTUAL BRAIDS

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ABSTRACT. We study combinatorial properties of virtual braid groups and we describe relations with finite type invariant theory for virtual knots and Yang Baxter equation.

## 1. INTRODUCTION

Virtual knots were introduced by L. Kauffman [Ka] as the geometric counterpart of Gauss diagrams. A virtual knot diagram is a generic (oriented) immersion of the circle into the plane, with the usual positive and negative crossings plus a new kind of crossings called virtual. The Gauss diagram of a virtual knot is constructed in the same way as for a classical knot excepted that virtual crossings are disregarded. Gauss diagrams turn to be in bijection with virtual knots diagrams up to isotopy and a finite number of “virtual” moves around crossings, which generalise usual Reidemeister moves. Using virtual Reidemeister moves we can introduce a notion of “virtual” braids (see for instance [B, K, V]). Virtual braids on  $n$  strands form a group, usually denoted by  $VB_n$ . The relations between virtual braids and virtual knots (and links) are completely determined by a generalisation of Alexander and Markov Theorems [K].

The paper is mainly devoted to the study of the kernels of two different projections of  $VB_n$  in  $S_n$ , the normal closure of the braid group  $B_n$ , that we will denote by  $H_n$ , and the so-called virtual pure braid group  $VP_n$ , which is related to the quantum Yang Baxter equation (see Section 5).

The paper is organised as follows: in Section 2 we recall some definitions and classical results on combinatorial group theory and in Section 3 we provide a short survey on lower central series for (generalised) braids and their relations with finite type invariant theory. In following Sections we determine the lower central series of the virtual braid group  $VB_n$  and of its subgroups  $H_n$  and  $VP_n$ . Finally, in Section 7 we determine the intersection of  $H_n$  and  $VP_n$  and in Section 8 we provide a connection between virtual pure braids and the finite type invariant theory for virtual knots defined by Goussarov, Polyak and Viro in [GPV].

## 2. DEFINITIONS

Given a group  $G$ , we define the *lower central series* of  $G$  as the filtration  $\Gamma_1(G) = G \supseteq \Gamma_2(G) \supseteq \dots$ , where  $\Gamma_i(G) = [\Gamma_{i-1}(G), G]$ . The *rational lower central series* of  $G$  is the filtration  $D_1(G) \supseteq D_2(G) \supseteq \dots$  obtained setting  $D_1(G) = G$ , and for  $i \geq 2$ , defining

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$D_i(G) = \{x \in G \mid x^n \in \Gamma_i(G) \text{ for some } n \in \mathbb{N}^*\}$ . This filtration was first considered by Stallings [S] and we denote it by the name proposed in [GL].

Let us recall that for any group-theoretic property  $\mathcal{P}$ , a group  $G$  is said to be *residually*  $\mathcal{P}$  if for any (non-trivial) element  $x \in G$ , there exists a group  $H$  with the property  $\mathcal{P}$  and a surjective homomorphism  $\phi : G \rightarrow H$  such that  $\phi(x) \neq 1$ . It is well known that a group  $G$  is residually nilpotent if and only if  $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$ . On the other hand, a group  $G$  is residually torsion-free nilpotent if and only if  $\bigcap_{i \geq 1} D_i(G) = \{1\}$ .

The fact that a group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable [MR]. We recall that a group  $G$  is said to be *bi-orderable* if there exists a strict total ordering  $<$  on its elements which is invariant under left and right multiplication, in other words,  $g < h$  implies that  $gk < hk$  and  $kg < kh$  for all  $g, h, k \in G$ .

Finally we recall that the *augmentation ideal* of a group  $G$  is the two-sided ideal  $I(G)$  of the group ring  $\mathbb{Z}[G]$  generated by the set  $\{g - 1 \mid g \in G\}$ . We denote by  $I^d(G)$  the  $d$ th power of  $I(G)$ .

The residually torsion free nilpotence of a group  $G$  implies that  $\bigcap_{d \geq 1} I^d(G) = \{1\}$  ([Pa], Theorem 2.15, chapter VI).

### 3. LOWER CENTRAL SERIES FOR GENERALIZED BRAID GROUPS

**Artin-Tits groups and surface braid groups.** Let us start by recalling some standard results on combinatorial properties of braid groups. It is well known (see [GoL] for instance) that the Artin braid group  $B_n$  is not residually nilpotent for  $n \geq 3$  and that the abelianization of  $B_n$  is isomorphic to  $\mathbb{Z}$  and  $\Gamma_2(B_n) = \Gamma_3(B_n)$ . We recall that classical braid groups are also called Artin-Tits groups of type  $\mathcal{A}$ . More precisely, let  $(W, S)$  be a Coxeter system and let us denote the order of the element  $st$  in  $W$  by  $m_{s,t}$  (for  $s, t \in S$ ). Let  $B_W$  be the group defined by the following group presentation:

$$B_W = \langle S \mid \underbrace{st \cdots}_{m_{s,t}} = \underbrace{ts \cdots}_{m_{s,t}} \text{ for any } s \neq t \in S \text{ with } m_{s,t} < +\infty \rangle.$$

The group  $B_W$  is the Artin-Tits group associated to  $W$ . The group  $B_W$  is said to be of spherical type if  $W$  is finite. The kernel of the canonical projection of  $B_W$  onto  $W$  is called the pure Artin-Tits group associated to  $W$ . As explained in [BGG], it is easy to show that the lower central series of almost all Artin-Tits groups of spherical type also stabilise at the second term, the only exception being Artin-Tits groups associated to the dihedral group  $I_{2m}$ , for  $m > 1$ .

When one considers surface braid groups (see [BGG] for a definition) new features appear. Let  $B_n(\Sigma)$  be the braid group on  $n$  strands on the surface  $\Sigma$ . In [BGG] it is proved that, when  $\Sigma$  is an oriented surface of positive genus and  $n \geq 3$ , the lower central series of  $B_n(\Sigma)$

stabilises at the third term. Moreover, the quotient groups associated to the lower central series form a complete (abelian) invariant for braid groups of closed surfaces.

**Pure braids and finite type invariants.** Let  $A, C$  be two groups. If  $C$  acts on  $A$  and the induced action on the abelianization of  $A$  is trivial, we say that  $A \rtimes C$  is an *almost-direct product* of  $A$  and  $C$ .

**Proposition 1.** ([FR]) *Let  $A, C$  be two groups. If  $C$  acts on  $A$  and the induced action on the abelianization of  $A$  is trivial, then*

$$I(A \rtimes C)^d = \sum_{k=0}^d I(A)^k \otimes I(C)^{d-k} \quad \text{for all } d \geq 0.$$

**Proposition 2.** *If  $A \rtimes C$  is an almost-direct product then  $\Gamma_m(A \rtimes C) = \Gamma_m(A) \rtimes \Gamma_m(C)$  and  $D_m(A \rtimes C) = D_m(A) \rtimes D_m(C)$ .*

*Proof.* The first statement is proved in [FR2]. Remark that the exact sequence  $1 \rightarrow A \rightarrow A \rtimes C \rightarrow C \rightarrow 1$  induces the following exact (splitting) sequence:

$$1 \rightarrow D_m(A \rtimes C) \cap A \rightarrow D_m(A \rtimes C) \rightarrow D_m(C) \rightarrow 1.$$

Therefore the second statement is equivalent to prove that  $D_m(A \rtimes C) \cap A = D_m(A)$  which is a straightforward consequence of the first statement and of the definition of rational lower central series.  $\square$

The structure of almost-direct product turns out to be a powerful tool in the determination of algebras related to lower central series (see for instance [CCP]) and more generally in the study of finite type invariants.

The pure braid group  $P_n$  is an almost-direct product of free groups [FR] and this fact has been used in [P] in order to construct an universal finite type invariant for braids with integers coefficients. In [GP] González-Meneses and Paris proved that the normal closure of the classical pure braid group  $P_n$  in the pure braid group on  $n$  strands of a surface  $\Sigma$  is an almost-direct product of (infinitely generated) free groups. Adapting the approach of Papadima, they constructed a universal finite type invariant for surface braids, but not multiplicative [BF].

**Remark 3.** *Free groups are residually torsion-free nilpotent [F, M]. It follows from Proposition 2 that pure braid groups are residually torsion-free nilpotent (see also [FR2]). More generally, using the faithfulness of the Krammer-Digne representation, Marin has shown recently that the pure Artin-Tits groups of spherical type are residually torsion-free nilpotent [Ma].*

#### 4. LOWER CENTRAL SERIES OF VIRTUAL BRAIDS

**Theorem 4.** ([V]) *The group  $VB_n$  admits the following group presentation:*

- **Generators:**  $\sigma_i, \rho_i, i = 1, 2, \dots, n-1$ .

• **Relations:**

$$(1) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n-2;$$

$$(2) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2;$$

$$(3) \quad \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \dots, n-2;$$

$$(4) \quad \rho_i \rho_j = \rho_j \rho_i, \quad |i - j| \geq 2;$$

$$(5) \quad \rho_i^2 = 1, \quad i = 1, 2, \dots, n-1;$$

$$(6) \quad \sigma_i \rho_j = \rho_j \sigma_i, \quad |i - j| \geq 2;$$

$$(7) \quad \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \dots, n-2.$$

**Remark 5.** Note that the last relation is equivalent to the following relation:

$$\rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i.$$

On the other hand, relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i$$

are not fulfilled in  $VB_n$  (see for instance [GPV]).

**Proposition 6.** Let  $\sigma_1, \dots, \sigma_{n-1}$  and respectively  $s_1, \dots, s_{n-1}$  be the usual generators of the Artin braid group  $B_n$  and of the symmetric group  $S_n$ . The morphism  $\iota : B_n \rightarrow VB_n$  defined by  $\iota(\sigma_i) = \sigma_i$  and the morphism  $\vartheta : S_n \rightarrow VB_n$  defined by  $\vartheta(s_i) = \rho_i$  are well defined and injective.

*Proof.* An easy argument for the injectivity of  $\iota$  is given in [K]. Another different proof is given in [R]. Now, let  $\mu : VB_n \rightarrow S_n$  be the morphism defined as follows (this morphism will be considered in Section 6):

$$\mu(\sigma_i) = 1, \quad \mu(\rho_i) = s_i, \quad i = 1, 2, \dots, n-1,$$

The set-section  $s : S_n \rightarrow VB_n$  defined by  $\mu(s_i) = \rho_i$  is a well defined morphism and thus the subgroup generated by  $\rho_1, \dots, \rho_{n-1}$  is isomorphic to  $S_n$  and  $\vartheta$  is injective.  $\square$

**Notation.** In the following we use the notations  $[a, b] = a^{-1}b^{-1}ab$  and  $a^b = b^{-1}ab$ .

**Proposition 7.** Let  $VB_n$  be the virtual braid group on  $n$  strands. The following properties hold:

- a) The group  $VB_2$  is isomorphic to  $\mathbb{Z} * \mathbb{Z}_2$  which is residually nilpotent.
- b) The group  $\Gamma_1(VB_n)/\Gamma_2(VB_n)$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_2$  for  $n \geq 2$ .
- c) The group  $\Gamma_2(VB_3)/\Gamma_3(VB_3)$  is isomorphic to  $\mathbb{Z}_2$ . Otherwise, if  $n \geq 4$  then  $\Gamma_2(VB_n) = \Gamma_3(VB_n)$ .
- d) If  $n \geq 3$  the group  $VB_n$  is not residually nilpotent.

*Proof.* a) The group  $\mathbb{Z} * \mathbb{Z}_2$  can be realised as a subgroup of  $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$  which is residually nilpotent (see [G] and [BGG]). It also follows from a result of Malcev [M].

- b) The statement can be easily verified considering  $VB_n$  provided with the group presentation given in Theorem 4.
- c) Consider the quotient  $G = VB_3/\Gamma_3(VB_3)$ . This group is generated by  $\bar{\sigma}_1 = \sigma_1\Gamma_3(VB_3)$ ,  $\bar{\sigma}_2 = \sigma_2\Gamma_3(VB_3)$ ,  $\bar{\rho}_1 = \rho_1\Gamma_3(VB_3)$  and  $\bar{\rho}_2 = \rho_2\Gamma_3(VB_3)$ . Since,

$$\sigma_2 = [\sigma_1, \sigma_2]\sigma_1, \quad \rho_2 = [\rho_1, \rho_2]\rho_1.$$

then  $\bar{\sigma}_2 = \bar{\sigma}_1$  and  $\bar{\rho}_2 = \bar{\rho}_1$  in  $G$ . So  $G = \langle \bar{\sigma}_1, \bar{\rho}_1 \rangle$  is 2-generated and 2-step nilpotent hence its commutator subgroup  $\Gamma_2(G)$  is cyclic. In  $\Gamma_2(G)$  the following relation is true  $[\bar{\sigma}_1, \bar{\rho}_1]^2 = [\bar{\sigma}_1, \bar{\rho}_1^2] = 1$ . Hence the commutator subgroup  $\Gamma_2(G)$  is generated by  $[\bar{\sigma}_1, \bar{\rho}_1]$  and has order  $\leq 2$ .

To see that  $[\bar{\sigma}_1, \bar{\rho}_1] \neq 1$  we recall that the unitriangular group  $UT_3(\mathbb{Z})$  over  $\mathbb{Z}$  is generated by two transvections,  $t_{12}(1)$  and  $t_{23}(1)$  (where  $t_{ij}(1) = e + e_{ij}$ ), and it is a free 2-step nilpotent group.

Now, there is a homomorphism  $\varphi$  of  $G$  onto the unitriangular group  $UT_3(\mathbb{Z}_2)$ , where  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ , by the rule  $\varphi(\bar{\sigma}_1) = t_{12}(\bar{1})$ ,  $\varphi(\bar{\rho}_1) = t_{23}(\bar{1})$ . It is easy to see that

$$\varphi([\bar{\sigma}_1, \bar{\rho}_1]) = [t_{12}(\bar{1}), t_{23}(\bar{1})] = t_{13}(\bar{1}) \neq e.$$

Hence,  $[\bar{\sigma}_1, \bar{\rho}_1] \neq 1$  in  $G$  and so  $\Gamma_2(VB_3)/\Gamma_3(VB_3) = \Gamma_2(G)$  is isomorphic to  $\mathbb{Z}_2$ .

Now, in order to prove the statement for  $n \geq 4$ , one can easily adapt an argument proposed in [BGG]. Denote  $\Gamma_i = \Gamma_i(VB_n)$  and consider the following short exact sequence:

$$1 \rightarrow \Gamma_2/\Gamma_3 \rightarrow \Gamma_1/\Gamma_3 \xrightarrow{p} \Gamma_1/\Gamma_2 \rightarrow 1,$$

Since any generator  $\sigma_i$  in  $\Gamma_1/\Gamma_3$  projects to the same element of  $\Gamma_1/\Gamma_2$ , for each  $1 \leq i \leq n-1$ , there exists  $t_i \in \Gamma_2/\Gamma_3$  (with  $t_1 = 1$ ) such that  $\sigma_i = t_i\sigma_1$ . Projecting the braid relation (1) into  $\Gamma_1/\Gamma_3$ , we see that  $t_i\sigma_1 t_{i+1}\sigma_1 t_i\sigma_1 = t_{i+1}\sigma_1 t_i\sigma_1 t_{i+1}\sigma_1$ . But the  $t_i$  are central in  $\Gamma_1/\Gamma_3$ , so  $t_i = t_{i+1}$ , and since  $t_1 = 1$ , we obtain  $\sigma_1 = \dots = \sigma_{n-1}$ . In the same way one obtains that  $\rho_1 = \dots = \rho_{n-1}$  in  $\Gamma_1/\Gamma_3$ . From relation (6) one deduces that  $\rho_1$  and  $\sigma_1$  commute in  $\Gamma_1/\Gamma_3$ . Therefore, the surjective homomorphism  $p$  is in fact an isomorphism.

- d) The group  $VB_n$  contains  $B_n$  (see Proposition 6) which is not residually nilpotent for  $n \geq 3$ . □

The group  $\Gamma_2(B_n)$  is perfect (i.e.  $\Gamma_2(B_n) = [\Gamma_2(B_n), \Gamma_2(B_n)]$ ) for  $n \geq 5$  [GoL]. An analogous result holds for the group  $\Gamma_2(VB_n)$ .

**Proposition 8.** *The group  $\Gamma_2(VB_n)$  is perfect for  $n \geq 5$ .*

*Proof.* Let  $A_n$  be the alternating group. The groups  $\Gamma_2(B_n)$  and  $\Gamma_2(S_n) = A_n$  are perfect for  $n \geq 5$  (moreover,  $A_n$  is simple for  $n \geq 5$ ). Consider a commutator  $[u, v] \in \Gamma_2(VB_n)$ . In order to prove the claim we need to show that  $[u, v] \in \Gamma_3(VB_n)$ . Since  $VB_n = \langle B_n, S_n \rangle$  we can use the following commutator identities

$$[ab, c] = [a, c]^b [b, c], \quad [a, bc] = [a, c][a, b]^c, \quad [a, b] = [b, a]^{-1}, \quad [a^{-1}, b] = [b, a]^{a^{-1}},$$

and we can represent the commutator  $[u, v]$  as a product of commutators

$$[\sigma_i, \sigma_j]^\alpha, \quad [\rho_i, \rho_j]^\beta, \quad [\sigma_i, \rho_j]^\gamma, \quad 1 \leq i, j \leq n-1, \quad \alpha, \beta, \gamma \in VB_n.$$

Since  $\Gamma_2(B_n)$  and  $\Gamma_2(S_n)$  are perfect for  $n \geq 5$ , then  $[\sigma_i, \sigma_j] \in [\Gamma_2(B_n), \Gamma_2(B_n)]$ ,  $[\rho_i, \rho_j] \in [\Gamma_2(S_n), \Gamma_2(S_n)]$ , and so  $[\sigma_i, \sigma_j]^\alpha$  and  $[\rho_i, \rho_j]^\beta$  belong to  $[\Gamma_2(VB_n), \Gamma_2(VB_n)]$ . Therefore we need only to prove that commutators of type  $[\sigma_i, \rho_j]$  belong to  $[\Gamma_2(VB_n), \Gamma_2(VB_n)]$ .

Consider  $[\sigma_i, \rho_j]$ . If  $|i - j| > 1$ , then  $[\sigma_i, \rho_j] = 1$ . Let  $|i - j| \leq 1$ . Then there are a pair  $k, l$ ,  $1 \leq k, l \leq n-1$ ,  $|k - l| > 1$  and two elements  $c_{i,k} \in \Gamma_2(B_n)$  and  $d_{j,l} \in \Gamma_2(S_n)$  such that  $\sigma_i = c_{i,k}\sigma_k$ ,  $\rho_j = d_{j,l}\rho_l$ . Hence,

$$[\sigma_i, \rho_j] = [c_{i,k}\sigma_k, d_{j,l}\rho_l].$$

Using commutator identities we have

$$\begin{aligned} [c_{i,k}\sigma_k, d_{j,l}\rho_l] &= [c_{i,k}\sigma_k, \rho_l][c_{i,k}\sigma_k, d_{j,l}]^{\rho_l} = [c_{i,k}, \rho_l]^{\sigma_k} [\sigma_k, \rho_l] ([c_{i,k}, d_{j,l}]^{\sigma_k} [\sigma_k, d_{j,l}])^{\rho_l} = \\ &= [c_{i,k}, \rho_l]^{\sigma_k} [c_{i,k}, d_{j,l}]^{\sigma_k \rho_l} [\sigma_k, d_{j,l}]^{\rho_l}. \end{aligned}$$

It is clear that  $[c_{i,k}, d_{j,l}] \in [\Gamma_2(VB_n), \Gamma_2(VB_n)]$  and therefore also  $[c_{i,k}, d_{j,l}]^{\sigma_k \rho_l}$  belongs to  $[\Gamma_2(VB_n), \Gamma_2(VB_n)]$ . Now, let us consider the commutator

$$[c_{i,k}, \rho_l]^{\sigma_k} = (c_{i,k}^{-1} c_{i,k}^{\rho_l})^{\sigma_k}.$$

Since  $c_{i,k}$  lies in  $\Gamma_2(B_n)$  and  $\Gamma_2(B_n)$  is perfect, then  $c_{i,k}$  lies in  $[\Gamma_2(B_n), \Gamma_2(B_n)]$  and hence  $[c_{i,k}, \rho_l]^{\sigma_k} \in [\Gamma_2(VB_n), \Gamma_2(VB_n)]$ . Analogously one can prove that the commutator  $[\sigma_k, d_{j,l}]^{\rho_l}$  belongs to  $[\Gamma_2(VB_n), \Gamma_2(VB_n)]$ . Hence  $\Gamma_2(VB_n)$  is perfect for  $n \geq 5$ .  $\square$

## 5. GENERATORS AND DEFINING RELATIONS OF THE VIRTUAL PURE BRAID GROUP $VP_n$

In this section we study the virtual pure braid group  $VP_n$ , introduced in [B]. Define the map

$$\nu : VB_n \longrightarrow S_n$$

of  $VB_n$  onto the symmetric group  $S_n$  as follows:

$$\nu(\sigma_i) = \nu(\rho_i) = \rho_i, \quad i = 1, 2, \dots, n-1,$$

where  $S_n$  is generated by  $\rho_i$  for  $i = 1, 2, \dots, n-1$ . The kernel  $\ker \nu$  is called the *virtual pure braid group on  $n$  strands* and it is denoted by  $VP_n$ .

Define the following elements

$$\lambda_{i,i+1} = \rho_i \sigma_i^{-1}, \quad \lambda_{i+1,i} = \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i, \quad i = 1, 2, \dots, n-1,$$

$$\lambda_{i,j} = \rho_{j-1} \rho_{j-2} \dots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \dots \rho_{j-2} \rho_{j-1},$$

$$\lambda_{j,i} = \rho_{j-1} \rho_{j-2} \dots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \dots \rho_{j-2} \rho_{j-1}, \quad 1 \leq i < j - 1 \leq n - 1.$$

**Theorem 9.** ([B]) *The group  $VP_n$  admits a presentation with the generators  $\lambda_{k,l}$ ,  $1 \leq k \neq l \leq n$ , and the defining relations:*

$$(8) \quad \lambda_{i,j} \lambda_{k,l} = \lambda_{k,l} \lambda_{i,j};$$

$$(9) \quad \lambda_{k,i} \lambda_{k,j} \lambda_{i,j} = \lambda_{i,j} \lambda_{k,j} \lambda_{k,i},$$

where distinct letters stand for distinct indices.

**Remark 10.** *It is worth to remark that the group  $VP_n$  has been independently defined and studied in [BE] in relation to Yang Baxter equations. More precisely, according to [BE], the virtual pure braid group on  $n$  strands is called the  $n$ -th quasitriangular group  $QTr_n$  and it is generated by  $R_{i,j}$  with  $1 \leq i \neq j \leq n$  with defining relations given by the quantum Yang Baxter equations*

$$\begin{aligned} R_{i,j} R_{k,l} &= R_{k,l} R_{i,j}; \\ R_{k,i} R_{k,j} R_{i,j} &= R_{i,j} R_{k,j} R_{k,i}. \end{aligned}$$

Since the natural section  $s : S_n \rightarrow VB_n$  is well defined one deduces that  $VB_n = VP_n \rtimes S_n$ . Moreover we can characterize the conjugacy action of  $S_n$  on  $VP_n$ . Let  $VP_n \rtimes S_n$  be the semidirect product defined by the action of  $S_n$  on the set  $\{\lambda_{k,l} \mid 1 \leq k \neq l \leq n\}$  by permutation of indices.

**Proposition 11.** ([B]) *The map  $\omega : VB_n \rightarrow VP_n \rtimes S_n$  sending any element  $v$  of  $VB_n$  into  $(v((s \circ \nu)(v))^{-1}, \nu(v)) \in VP_n \rtimes S_n$  is an isomorphism.*

Let us define the subgroup

$$V_i = \langle \lambda_{1,i+1}, \lambda_{2,i+1}, \dots, \lambda_{i,i+1}; \lambda_{i+1,1}, \lambda_{i+1,2}, \dots, \lambda_{i+1,i} \rangle, \quad i = 1, \dots, n-1,$$

of  $VP_n$ . Let  $V_i^*$  be the normal closure of  $V_i$  in  $VP_n$ . We have a “forgetting map”

$$\varphi : VP_n \longrightarrow VP_{n-1}$$

which takes generators  $\lambda_{i,n}$ ,  $i = 1, 2, \dots, n-1$ , and  $\lambda_{n,i}$ ,  $i = 1, 2, \dots, n-1$ , to the unit and fixes other generators. The kernel of  $\varphi$  is the group  $V_{n-1}^*$ , which turns out to be a free group infinitely generated.

**Theorem 12.** ([B]) *The group  $VP_n$ ,  $n \geq 2$ , is representable as the semi-direct product*

$$VP_n = V_{n-1}^* \rtimes VP_{n-1} = V_{n-1}^* \rtimes (V_{n-2}^* \rtimes (\dots \rtimes (V_2^* \rtimes V_1^*) \dots)),$$

where  $V_1^*$  is a free group of rank 2 and  $V_i^*$ ,  $i = 2, 3, \dots, n-1$ , are free groups infinitely generated.

The group  $V_{n-1}^*$  is the normal closure of the set  $\{\lambda_{1,n}, \lambda_{2,n}, \dots, \lambda_{n-1,n}, \lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,n-1}\}$ . We refer to [B] for a (infinite) free family of generators. In the following we set  $a^{\pm b}$  for  $b^{-1}a^{\pm 1}b$ .

**Lemma 13.** ([B]) *The following formulae are fulfilled in the group  $VP_n$ :*

- 1)  $\lambda_{n,l}^{\varepsilon} = \lambda_{n,l}, \max\{i, j\} < \max\{n, l\}, \varepsilon = \pm 1$ ;
- 2)  $\lambda_{i,n}^{\lambda_{i,j}} = \lambda_{n,j}^{\lambda_{i,j}} \lambda_{i,n} \lambda_{n,j}^{-1}, \lambda_{i,n}^{\lambda_{i,j}^{-1}} = \lambda_{n,j}^{-1} \lambda_{i,n} \lambda_{n,j}^{\lambda_{i,j}^{-1}}, i < j < n \text{ or } j < i < n$ ;
- 3)  $\lambda_{n,i}^{\lambda_{i,j}} = \lambda_{n,j} \lambda_{n,i} \lambda_{n,j}^{-\lambda_{i,j}}, \lambda_{n,i}^{\lambda_{i,j}^{-1}} = \lambda_{n,j}^{-\lambda_{i,j}^{-1}} \lambda_{n,i} \lambda_{n,j}, i < j < n \text{ or } j < i < n$ ;
- 4)  $\lambda_{j,n}^{\lambda_{i,j}} = \lambda_{i,n} \lambda_{j,n} \lambda_{n,j} \lambda_{i,n}^{-1} \lambda_{n,j}^{-\lambda_{i,j}}, \lambda_{j,n}^{\lambda_{i,j}^{-1}} = \lambda_{j,n}^{-\lambda_{i,j}^{-1}} \lambda_{i,n} \lambda_{n,j} \lambda_{i,j}, i < j < n \text{ or } j < i < n$ ,

where different letters stand for different indices.

Lemma 13 provides the action of  $VP_{n-1}$  on  $V_{n-1}^*$ . Therefore one can deduce that the semi-direct product given in Theorem 12 fails to be an almost-direct product.

Nevertheless we can give a partial result on lower central series of virtual pure braids. Let us start considering the group  $VP_3$ . It is generated by the elements

$$\lambda_{2,1}, \lambda_{1,2}, \lambda_{3,1}, \lambda_{3,2}, \lambda_{2,3}, \lambda_{1,3},$$

and defined by relations

$$\lambda_{1,2}(\lambda_{1,3}\lambda_{2,3}) = (\lambda_{2,3}\lambda_{1,3})\lambda_{1,2}, \lambda_{2,1}(\lambda_{2,3}\lambda_{1,3}) = (\lambda_{1,3}\lambda_{2,3})\lambda_{2,1},$$

$$\lambda_{3,1}(\lambda_{3,2}\lambda_{1,2}) = (\lambda_{1,2}\lambda_{3,2})\lambda_{3,1}, \lambda_{3,2}(\lambda_{3,1}\lambda_{2,1}) = (\lambda_{2,1}\lambda_{3,1})\lambda_{3,2},$$

$$\lambda_{2,3}(\lambda_{2,1}\lambda_{3,1}) = (\lambda_{3,1}\lambda_{2,1})\lambda_{2,3}, \lambda_{1,3}(\lambda_{1,2}\lambda_{3,2}) = (\lambda_{3,2}\lambda_{1,2})\lambda_{1,3}.$$

Define the following order on the generators:

$$\lambda_{1,2} < \lambda_{2,1} < \lambda_{1,3} < \lambda_{2,3} < \lambda_{3,1} < \lambda_{3,2}.$$

With this order we can consider the following 15 commutators as the basic commutators of  $VP_3/\Gamma_3(VP_3)$ .

$$[\lambda_{3,2}, \lambda_{3,1}], [\lambda_{3,2}, \lambda_{2,3}], [\lambda_{3,2}, \lambda_{1,3}], [\lambda_{3,2}, \lambda_{2,1}], [\lambda_{3,2}, \lambda_{1,2}],$$

$$[\lambda_{3,1}, \lambda_{2,3}], [\lambda_{3,1}, \lambda_{1,3}], [\lambda_{3,1}, \lambda_{2,1}], [\lambda_{3,1}, \lambda_{1,2}],$$

$$[\lambda_{2,3}, \lambda_{1,3}], [\lambda_{2,3}, \lambda_{2,1}], [\lambda_{2,3}, \lambda_{1,2}],$$

$$[\lambda_{1,3}, \lambda_{2,1}], [\lambda_{1,3}, \lambda_{1,2}],$$

$$[\lambda_{2,1}, \lambda_{1,2}].$$



In  $VP_3/\Gamma_3(VP_3)$  the defining relations will have the following form (relations are written in terms of basic commutators):

- 1)  $[\lambda_{2,3}, \lambda_{1,3}][\lambda_{2,3}, \lambda_{1,2}][\lambda_{1,3}, \lambda_{1,2}] = 1,$
- 2)  $[\lambda_{2,3}, \lambda_{1,3}]^{-1}[\lambda_{1,3}, \lambda_{2,1}][\lambda_{2,3}, \lambda_{2,1}] = 1,$
- 3)  $[\lambda_{3,2}, \lambda_{3,1}]^{-1}[\lambda_{3,1}, \lambda_{1,2}][\lambda_{3,2}, \lambda_{1,2}] = 1,$
- 4)  $[\lambda_{3,2}, \lambda_{3,1}][\lambda_{3,2}, \lambda_{2,1}][\lambda_{3,1}, \lambda_{2,1}] = 1,$
- 5)  $[\lambda_{3,1}, \lambda_{2,3}]^{-1}[\lambda_{3,1}, \lambda_{2,1}]^{-1}[\lambda_{2,3}, \lambda_{2,1}] = 1,$
- 6)  $[\lambda_{3,2}, \lambda_{1,2}][\lambda_{3,2}, \lambda_{1,3}][\lambda_{1,3}, \lambda_{1,2}]^{-1} = 1.$

We see that each from the following commutators

$$[\lambda_{2,3}, \lambda_{1,2}], [\lambda_{1,3}, \lambda_{2,1}], [\lambda_{3,1}, \lambda_{1,2}], [\lambda_{3,2}, \lambda_{2,1}], [\lambda_{3,1}, \lambda_{2,3}], [\lambda_{3,2}, \lambda_{1,3}],$$

is included only once in the list of relations 1) - 6). Hence, we can exclude these commutators. Then we get the following result.

**Lemma 14.**  $\Gamma_2(VP_3)/\Gamma_3(VP_3) \simeq \mathbb{Z}^9$ .

Now, let us outline the general case. The group  $VP_n$  contains  $n(n-1)$  generators. Define the following order on the generators:

$$\begin{aligned} \lambda_{1,2} < \lambda_{2,1} < \lambda_{1,3} < \lambda_{2,3} < \lambda_{3,1} < \lambda_{3,2} < \dots < \lambda_{1,n} < \lambda_{2,n} < \dots < \\ < \lambda_{n-1,n} < \lambda_{n,1} < \lambda_{n,2} < \dots < \lambda_{n,n-1}. \end{aligned}$$

The group  $\Gamma_2(VP_n)/\Gamma_3(VP_n)$  is generated by  $M = n(n-1)(n^2 - n - 1)/2$  of the basic commutators  $[\lambda_{i,j}, \lambda_{k,l}]$  with  $\lambda_{i,j} > \lambda_{k,l}$ .

There are two types of relations in  $VP_n$ :

$$\begin{aligned} \lambda_{i,j}\lambda_{k,l} &= \lambda_{k,l}\lambda_{i,j}, \\ \lambda_{k,i}\lambda_{k,j}\lambda_{i,j} &= \lambda_{i,j}\lambda_{k,j}\lambda_{k,i}. \end{aligned}$$

The number of relations of the first type is equal to

$$n(n-1)(n-2)(n-3).$$

Hence, the number of basic commutators which are trivial in  $\Gamma_2(VP_n)/\Gamma_3(VP_n)$  is equal to

$$M_1 = n(n-1)(n-2)(n-3)/2.$$

The number of relations of the second type is equal to

$$M_2 = n(n-1)(n-2).$$

The relation of the second type modulo  $\Gamma_3(VP_n)$  has the form

$$[\lambda_{i,j}, \lambda_{k,j}][\lambda_{i,j}, \lambda_{k,i}][\lambda_{k,j}, \lambda_{k,i}] = 1.$$

Each commutator in this relation is basic or inverse to basic commutators. As in the case  $n = 3$  one can check that the commutator  $[\lambda_{i,j}, \lambda_{k,i}]$  itself and its inverse (i.e.  $[\lambda_{k,i}, \lambda_{i,j}]$ ) don't include in other relations. Therefore we can exclude from basic commutators  $[\lambda_{i,j}, \lambda_{k,i}]$  and its inverse. Hence, the number of non-trivial basic commutators in  $\Gamma_2(VP_n)/\Gamma_3(VP_n)$  is equal to  $M - M_1 - M_2$ , and we have the following result.

**Proposition 15.** *The group  $\Gamma_2(VP_n)/\Gamma_3(VP_n)$ , for  $n \geq 3$ , is a free abelian group of rank  $n(n-1)(2n-3)/2$ .*

The group  $VP_2$  is free of rank 2. Remark also that Proposition 15 implies that for  $n \geq 3$  the quotient  $D_2(VP_n)/D_3(VP_n)$  is a free abelian group of rank  $n(n-1)(2n-3)/2$  since for any group  $G$ ,  $\Gamma_i(G)/\Gamma_{i+1}(G)$  is isomorphic to  $D_i(G)/D_{i+1}(G)$  modulo torsion (see for instance [H]).

**Question.** The group  $VP_n$  is residually torsion-free nilpotent?

## 6. THE GROUP $H_n$

In this section we prove another decomposition of  $VB_n$  as a semidirect product. Let  $\mu : VB_n \rightarrow S_n$  be the morphism defined as follows:

$$\mu(\sigma_i) = 1, \mu(\rho_i) = \rho_i, i = 1, 2, \dots, n-1,$$

where  $S_n$  is generated by  $\rho_i$  for  $i = 1, 2, \dots, n-1$ . Let us denote by  $H_n$  the normal closure of  $B_n$  in  $VB_n$ .

It is evident that  $\ker \mu$  coincides with  $H_n$ . Now, define the following elements:

$$x_{i,i+1} = \sigma_i, x_{i,j} = \rho_{j-1} \cdots \rho_{i+1} \sigma_i \rho_{i+1} \cdots \rho_{j-1} \text{ for } 1 \leq i < j-1 \leq n-1,$$

$$x_{i+1,i} = \rho_i \sigma_i \rho_i \text{ and } x_{j,i} = \rho_{j-1} \cdots \rho_{i+1} \rho_i \sigma_i \rho_i \rho_{i+1} \cdots \rho_{j-1} \text{ for } 1 \leq i < j-1 \leq n-1.$$

Since the subgroup of  $VB_n$  generated by  $\rho_1, \dots, \rho_{n-1}$  is isomorphic to the symmetric group  $S_n$  (Proposition 6), we can define an action of  $S_n = \langle \rho_1, \dots, \rho_{n-1} \rangle$  on the set  $\{x_{i,j}, 1 \leq i \neq j \leq n\}$  by permutation of indices, i. e.,  $x_{i,j}^\rho = x_{\rho(i), \rho(j)}$ ,  $\rho \in S_n$ .

**Lemma 16.** *Let  $\rho \in S_n$ . The element  $\rho x_{i,j} \rho^{-1}$  is equivalent to  $x_{\rho(i), \rho(j)}$  for  $1 \leq i \neq j \leq n-1$ .*

*Proof.* It is sufficient to prove the statement only for generators  $\rho_k$ , for  $1 \leq k \leq n-1$ . If  $\rho_k \neq \rho_i, \rho_{i+1}, \rho_{i-1}$  from relation (6) in Theorem 4 one obtains that  $\rho_k \sigma_i \rho_k = \sigma_i = x_{i,i+1} = x_{\rho(i), \rho(i+1)}$ . Otherwise  $\rho_{i+1} \sigma_i \rho_{i+1} = x_{i,i+2}$ ,  $\rho_i \sigma_i \rho_i = x_{i+1,i}$  or, by relation (7) in Theorem 4,  $\rho_{i-1} \sigma_i \rho_{i-1} = \rho_i \sigma_{i-1} \rho_i = x_{i-1,i}$ .

Let  $x_{i,j} = \rho_{j-1} \cdots \rho_{i+1} \sigma_i \rho_{i+1} \cdots \rho_{j-1}$  for  $1 \leq i < j-1 \leq n-1$ .

- i) Let  $k > j$  or  $k < i-1$ . Then  $\rho_k$  commute with  $x_{i,j}$  and the claim holds.
- ii) If  $k = j$ , by definition,  $\rho_j x_{i,j} \rho_j = x_{i,j+1}$ . Since  $\rho_j(i) = i$  and  $\rho_j(j) = j+1$  the claim holds.
- iii) If  $k = j-1$  one deduces that  $\rho_{j-1} x_{i,j} \rho_{j-1} = x_{i,j-1} = x_{\rho_{j-1}(i), \rho_{j-1}(j-1)}$ .
- iv) If  $i < k < j-1$  it suffices to remark that  $\rho_k \rho_{j-1} \cdots \rho_{i+1} = \rho_{j-1} \cdots \rho_{i+1} \rho_{k+1}$  and that  $\rho_{k+1} \rho_{i+1} \cdots \rho_{j-1} = \rho_{i+1} \cdots \rho_{j-1} \rho_k$ . Therefore  $\rho_k x_{i,j} \rho_k^{-1} = x_{i,j} = x_{\rho_k(i), \rho_k(j)}$ .

v) If  $k = i$ , we remark first that the equality

$$\rho_i \rho_{j-1} \cdots \rho_{i+1} \sigma_i \rho_{i+1} \cdots \rho_{j-1} \rho_i = \rho_{j-1} \cdots \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \cdots \rho_{j-1}$$

holds in  $VB_n$ . Applying relations (4) and (7) of Theorem 4 and we obtain the following equality:

$$\rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i = \sigma_{i+1},$$

and then  $\rho_i x_{i,j} \rho_i = x_{i+1,j} = x_{\rho(i), \rho(i+1)}$ .

vi) If  $k = i - 1$  it suffices to remark that applying relations (4) and (7) of Theorem 4 one obtains the following equalities:

$$\begin{aligned} \rho_{i-1} \sigma_i \rho_{i-1} &= \rho_i \rho_i \rho_{i-1} \sigma_i \rho_{i-1} = \\ \rho_i \sigma_{i-1} \rho_i \rho_{i-1} \rho_{i-1} &= \rho_i \sigma_{i-1} \rho_i. \end{aligned}$$

The case of  $x_{i+1,i}$  and  $x_{j,i}$  for  $1 \leq i < j - 1 \leq n - 1$  are similar and they are left to the reader.  $\square$

**Proposition 17.** *The group  $H_n$  admits a presentation with the generators  $x_{k,l}$ ,  $1 \leq k \neq l \leq n$ , and the defining relations:*

$$(10) \quad x_{i,j} x_{k,l} = x_{k,l} x_{i,j},$$

$$(11) \quad x_{i,k} x_{k,j} x_{i,k} = x_{k,j} x_{i,k} x_{k,j},$$

where distinct letters stand for distinct indices.

*Proof.* We use the Reidemeister–Schreier method (see, for example, Chapter 2.2 of [KMS]). The set of elements of  $\langle \rho_1, \dots, \rho_{n-1} \rangle$  in normal form:

$$\begin{aligned} \Lambda_n = \{ & (\rho_{i_1} \rho_{i_1-1} \cdots \rho_{i_1-r_1}) (\rho_{i_2} \rho_{i_2-1} \cdots \rho_{i_2-r_2}) \cdots (\rho_{i_p} \rho_{i_p-1} \cdots \rho_{i_p-r_p}) \mid \\ & 1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, 0 \leq r_j < i_j \} \end{aligned}$$

is a Schreier set of coset representatives of  $H_n$  in  $VB_n$ . Define the map  $^- : VB_n \longrightarrow \Lambda_n$  which takes an element  $w \in VB_n$  into the representative  $\bar{w}$  from  $\Lambda_n$ . The element  $w \bar{w}^{-1}$  belongs to  $H_n$ . The group  $H_n$  is generated by

$$s_{\rho,a} = \rho a \cdot (\bar{\rho a})^{-1},$$

where  $\rho$  runs over the set  $\Lambda_n$  and  $a$  runs over the set of generators of  $VB_n$  (Theorem 2.7 of [KMS]).

It is easy to establish that  $s_{\rho,\rho_i} = e$  for any  $\rho \in \Lambda_n$  and any generator  $\rho_i$  of  $VB_n$ .

On the other hand,

$$s_{\rho,\sigma_i} = \rho \sigma_i \cdot (\bar{\rho \sigma_i})^{-1} = \rho \sigma_i \cdot (\bar{\rho})^{-1} = \rho \sigma_i \rho^{-1} = x_{\rho(i), \rho(i+1)}.$$

It follows that each generator  $s_{\rho,\sigma_i}$  is equal to some  $x_{i,j}$ ,  $1 \leq i \neq j \leq n$ . The inverse statement is also true, i. e., each element  $x_{i,j}$  is equal to some generator  $s_{\rho,\sigma_i}$ .

To find defining relations of  $H_n$  we define a rewriting process  $\tau$ . It allows us to rewrite a word which is written in the generators of  $VB_n$  and presents an element in  $H_n$  as a word in the generators of  $H_n$ . Let us associate to the reduced word

$$u = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_\nu^{\varepsilon_\nu}, \quad \varepsilon_l = \pm 1, \quad a_l \in \{\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho_1, \rho_2, \dots, \rho_{n-1}\},$$

the word

$$\tau(u) = s_{k_1, a_1}^{\varepsilon_1} s_{k_2, a_2}^{\varepsilon_2} \dots s_{k_\nu, a_\nu}^{\varepsilon_\nu}$$

in the generators of  $H_n$ , where  $k_j$  is a representative of the  $(j-1)$ th initial segment of the word  $u$  if  $\varepsilon_j = 1$  and  $k_j$  is a representative of the  $j$ th initial segment of the word  $u$  if  $\varepsilon_j = -1$ .

The group  $H_n$  is defined by relations

$$r_{\mu, \rho} = \tau(\rho r_\mu \rho^{-1}), \quad \rho \in \Lambda_n,$$

where  $r_\mu$  is the defining relation of  $VB_n$  (Theorem 2.9 of [KMS]). Denote by

$$r_1 = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$$

the first relation of  $VB_n$ . Then

$$\begin{aligned} r_{1,e} &= \tau(r_1) = s_{e, \sigma_i} s_{\overline{\sigma_i}, \sigma_{i+1}} s_{\overline{\sigma_i \sigma_{i+1}}, \sigma_i} s_{\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1}, \sigma_{i+1}}^{-1} s_{\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1}, \sigma_i}^{-1} s_{\overline{r_1}, \sigma_{i+1}}^{-1} = \\ &= s_{e, \sigma_i} s_{e, \sigma_{i+1}} s_{e, \sigma_i} s_{e, \sigma_{i+1}}^{-1} s_{e, \sigma_i}^{-1} s_{e, \sigma_{i+1}}^{-1} = x_{i, i+1} x_{i+1, i+2} x_{i, i+1} x_{i+1, i+2}^{-1} x_{i, i+1}^{-1} x_{i+1, i+2}^{-1}. \end{aligned}$$

Therefore, the following relation

$$x_{i, i+1} x_{i+1, i+2} x_{i, i+1} = x_{i+1, i+2} x_{i, i+1} x_{i+1, i+2}$$

is fulfilled in  $H_n$ . The remaining relations  $r_{1, \rho}$ ,  $\rho \in \Lambda_n$ , can be obtained from this relation using conjugation by  $\rho$ :

$$\begin{aligned} r_{1, \rho} &= \rho x_{i, i+1} x_{i+1, i+2} x_{i, i+1} x_{i+1, i+2}^{-1} x_{i, i+1}^{-1} x_{i+1, i+2}^{-1} \rho^{-1} = \\ &= x_{\rho(i), \rho(i+1)} x_{\rho(i+1), \rho(i+2)} x_{\rho(i), \rho(i+1)} x_{\rho(i+1), \rho(i+2)}^{-1} x_{\rho(i), \rho(i+1)}^{-1} x_{\rho(i+1), \rho(i+2)}^{-1} \end{aligned}$$

and we obtain relations (11).

Let us consider the next relation of  $VB_n$ :

$$r_2 = \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}, \quad |i - j| \geq 2.$$

Applying the rewriting process defined above we obtain that following equality holds in  $H_n$ :

$$r_{2,e} = \tau(r_2) = s_{e, \sigma_i} s_{\overline{\sigma_i}, \sigma_j} s_{\sigma_i \sigma_j \sigma_i^{-1}, \sigma_i}^{-1} s_{\overline{r_2}, \sigma_j}^{-1} = x_{i, i+1}^{-1} x_{j, j+1}^{-1} x_{i, i+1} x_{j, j+1}.$$

Conjugating this relation by all representatives from  $\Lambda_n$  and applying Lemma 16 as above, we obtain relations (10).

Let us prove that only trivial relations follow from all other relations of  $VB_n$ . It is evident for relations (3)–(5) defining the group  $S_n$  because  $s_{\rho, \rho_i} = e$  for all  $\rho \in \Lambda_n$  and  $\rho_i$ .

Consider the mixed relation (7) (relation (6) can be considered similarly):

$$r_3 = \sigma_{i+1} \rho_i \rho_{i+1} \sigma_i^{-1} \rho_{i+1} \rho_i.$$

Using the rewriting process, we get

$$\begin{aligned} r_{3,e} &= \tau(r_3) = s_{e,\sigma_{i+1}} \frac{s^{-1}}{\sigma_{i+1}\rho_i\rho_{i+1}\sigma_i^{-1},\sigma_i} = \\ &= x_{i+1,i+2}(\rho_i \rho_{i+1} x_{i,i+1}^{-1} \rho_{i+1} \rho_i) = e. \end{aligned}$$

□

The following Corollary is a straightforward consequence of Lemma 16 and of the fact that the natural section  $S_n \rightarrow VB_n$  is well defined.

**Corollary 18.** *The group  $VB_n$  is isomorphic to  $H_n \rtimes S_n$  where  $S_n$  acts by permutation of indices.*

From Proposition 17 we derive that  $H_n$ , which is the normal closure of  $B_n$ , is an Artin-Tits group, but not of spherical type. It is also easy to verify that the lower central series of  $H_n$  is similar to the one of  $B_n$ .

**Proposition 19.** *The following properties hold:*

- a) *The group  $H_2$  is isomorphic to  $\mathbb{Z} * \mathbb{Z}$  which is residually nilpotent.*
- b) *The quotient  $\Gamma_1(H_3)/\Gamma_2(H_3)$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  and if  $n \geq 4$  then  $\Gamma_1(H_n)/\Gamma_2(H_n)$  is isomorphic to  $\mathbb{Z}$ .*
- c) *If  $n \geq 3$  then the group  $H_n$  is not residually nilpotent and  $\Gamma_2(H_n) = \Gamma_3(H_n)$ .*
- d) *If  $n \geq 5$  the group  $\Gamma_2(H_n)$  is perfect.*

*Proof.* We prove the point b). If  $n = 3$ , from the six defining relations one deduces that  $x_{1,2} = x_{2,3} = x_{3,1}$  and  $x_{1,3} = x_{3,2} = x_{2,1}$  in  $\Gamma_1(H_3)/\Gamma_2(H_3)$  which turns to be isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ . When  $n \geq 4$ , given two elements  $x_{i,j}$  and  $x_{k,l}$  we have the following cases:

- i) If  $j = k$  and  $i \neq l$  from the relation  $x_{i,j}x_{j,l}x_{i,j} = x_{j,l}x_{i,j}x_{j,l}$  one deduces that  $x_{i,j} = x_{j,l}$  in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .
- ii) If  $j \neq k$  and  $i = l$  we conclude as above that  $x_{k,i} = x_{i,j}$  in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .
- iii) If  $j \neq l$  and  $i = k$  there exists  $1 \leq m \leq n$  distinct from  $j, l, k$  such that  $x_{k,j} = x_{j,m} = x_{m,k} = x_{k,l}$  in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .
- iv) If  $i \neq k$  and  $j = l$  we proceed as in previous case and we obtain that  $x_{i,j} = x_{k,j}$  in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .
- v) If  $i, j, k, l$  are distinct, using the element  $x_{j,k}$  it is clear that  $x_{i,j} = x_{k,l}$  in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .
- vi) Finally if  $i = l$  and  $j = k$ , we choose  $1 \leq m, p \leq n$  distinct from  $i$  and  $j$  and we obtain the following sequence of identities

$$x_{i,j} = x_{j,m} = x_{m,p} = x_{p,j} = x_{j,i}$$

holds in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .

Therefore all  $x_{i,j}$  are identified in  $\Gamma_1(H_n)/\Gamma_2(H_n)$ .

To prove c) and d) we recall that  $H_n$  is the normal closure of  $B_n$  and therefore is not residually nilpotent for  $n \geq 3$ . Moreover, we recall that from the Artin braid relations, it follows that  $\Gamma_2(B_n)$  is the normal closure in  $B_n$  of the element  $\sigma_1\sigma_2^{-1}$  (see for instance [BGG]), and thus  $\Gamma_2(H_n)$  coincides with the normal closure in  $VB_n$  of the element  $\sigma_1\sigma_2^{-1}$ . Since

$\sigma_1\sigma_2^{-1} = [[\sigma_1, \sigma_2], \sigma_1]^{\sigma_1}$  in  $B_n$ , then  $\sigma_1\sigma_2^{-1} = [[\sigma_1, \sigma_2], \sigma_1]^{\sigma_1}$  in  $VB_n$  and therefore  $\Gamma_2(H_n) = \Gamma_3(H_n)$ . In the same way, since  $\Gamma_2(B_n)$  is perfect for all  $n \geq 5$  [GL], so is  $\Gamma_2(H_n)$ .  $\square$

**Remark 20.** *The group  $H_3$  decomposes as a free product  $G_1 * G_2$ , where  $G_1 = \langle x_{1,2}, x_{2,3}, x_{3,1} \rangle$ ,  $G_2 = \langle x_{1,3}, x_{3,2}, x_{2,1} \rangle$ . The group  $G_i$ ,  $i = 1, 2$  is isomorphic to the 2nd affine Artin-Tits group of type  $\mathcal{A}$ , also called circular braid group on 3 strands (see [AJ]).*

The decomposition of  $VB_n$  into semidirect product provided in Corollary 18 was earlier proposed by Rabenda ([R]). More precisely let  $K_n$  be the (abstract) group with the following group presentation:

- Generators:  $x_{i,j}$  for  $1 \leq i \neq j \leq n$ .
- Relations:  $x_{i,j} x_{j,k} x_{i,j} = x_{j,k} x_{i,j} x_{j,k}$  for  $i, j, k$  distinct indices.

The symmetric group  $S_n$  acts transitively on  $K_n$  by permutation of indices: for any  $\sigma$  in  $S_n$ ,  $x_{i,j}^\sigma = x_{\sigma(i), \sigma(j)}$ . Let  $G_n$  be the semi-direct product of  $K_n$  and  $S_n$  defined by above action and let  $s_1, \dots, s_n$  be the generators of  $S_n$  considered as generators of  $G_n$ .

Rabenda defined a map  $\phi : G_n \rightarrow VB_n$  as follows;  $\phi(s_i) = \rho_i$  for  $i = 1, \dots, n-1$ ,  $\phi(x_{i,i+1}) = \sigma_i$ ,  $\phi(x_{i,j}) = \rho_{j-1} \cdots \rho_{i+1} \sigma_i \rho_{i+1} \cdots \rho_{j-1}$  for  $1 \leq i < j-1 \leq n-1$ ,  $\phi(x_{i+1,i}) = \rho_i \sigma_i \rho_i$  and  $\phi(x_{j,i}) = \rho_{j-1} \cdots \rho_{i+1} \rho_i \sigma_i \rho_i \rho_{i+1} \cdots \rho_{j-1}$  for  $1 \leq i < j-1 \leq n-1$  and he outlined a proof of the fact that the morphism  $\phi$  is actually an isomorphism.

## 7. THE EXTENDED PURE BRAID GROUP $EP_n$

In this section we determine the relations between the group  $H_n$  and the group  $VP_n$ .

**Proposition 21.** *The group  $H_n$  and  $VP_n$  are not isomorphic for  $n \geq 3$ .*

*Proof.* It suffices to remark that the abelianisation of  $VP_n$  is isomorphic to  $\mathbb{Z}^{n(n-1)}$  and to compare with the abelianisation of  $H_n$  (part b) of Proposition 19).  $\square$

Let  $\varphi$  be the map  $S_n \rightarrow \text{Hom}(S_n)$  defined by the action of the symmetric group on itself by conjugacy.

**Proposition 22.** *The semidirect product  $S_n \rtimes_\varphi S_n$  admits the following group presentation:*

- **Generators:**  $s_i, t_i$ ,  $i = 1, 2, \dots, n-1$
- **Relations:**

$$\begin{aligned}
 t_i^2 &= s_i^2 = 1, \quad i = 1, 2, \dots, n-1 \\
 s_i s_j &= s_j s_i, \quad |i-j| \geq 2 \\
 s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \quad i = 1, 2, \dots, n-2 \\
 t_i t_j &= t_j t_i, \quad |i-j| \geq 2 \\
 t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, \quad i = 1, 2, \dots, n-2 \\
 t_i s_j t_i &= s_j, \quad |i-j| \geq 2 \\
 t_i s_i t_i &= s_i, \quad i = 1, 2, \dots, n-1 \\
 t_{i+1} s_i t_{i+1} &= s_{i+1} s_i s_{i+1}, \quad i = 1, 2, \dots, n-2 \\
 t_{i-1} s_i t_{i-1} &= s_{i-1} s_i s_{i-1}, \quad i = 1, 2, \dots, n-1
 \end{aligned}$$

In the following we set  $M_n$  the semidirect product  $S_n \rtimes_{\varphi} S_n$ . Let  $VB_n$  and  $M_n$  be provided with the group presentations given respectively in Theorem 4 and in Proposition 22 and let  $\chi : VB_n \rightarrow M_n$  be the morphism defined by  $\chi(\sigma_i) = s_i$  and  $\chi(\rho_i) = t_i$  for  $i = 1, 2, \dots, n-1$ . We call *extended pure braid group on  $n$  strands* the kernel  $\ker \chi$  and we denote it by  $EP_n$ .

Let  $\eta_1$  and  $\eta_2$  be the maps from  $M_n$  to  $S_n$  defined respectively as follows:

$$\eta_1(s_i) = 1 \text{ and } \eta_1(t_i) = \rho_i \text{ for } i = 1, 2, \dots, n-1 ;$$

$$\eta_2(s_i) = \rho_i \text{ and } \eta_2(t_i) = \rho_i \text{ for } i = 1, 2, \dots, n-1 ,$$

where  $\rho_i$ , for  $i = 1, 2, \dots, n-1$ , are the usual generators of  $S_n$ .

**Proposition 23.** *Let  $VP_n$  and  $H_n$  be provided with the group presentations given respectively in Theorem 9 and Proposition 17.*

- i) *The group  $H_n$  coincides with  $\ker(\eta_1 \circ \chi)$ .*
- ii) *The group  $VP_n$  coincides with  $\ker(\eta_2 \circ \chi)$ .*
- iii) *The group  $H_n \cap VP_n$  coincides with  $\ker \chi$ .*

*Proof.* We recall that  $\mu(\sigma_i) = 1$ ,  $\mu(\rho_i) = \rho_i$ ,  $\nu(\sigma_i) = \rho_i$  and  $\nu(\rho_i) = \rho_i$  for  $i = 1, 2, \dots, n-1$ . Therefore  $\mu = \eta_1 \circ \chi$  and  $\nu = \eta_2 \circ \chi$  and part i) and ii) follow and then  $\ker \chi \subseteq H_n \cap VP_n$ . Now let  $x$  be a non trivial element of  $H_n$ . Since  $H_n$  is the normal closure of  $B_n$ , the element  $\chi(x)$  belongs to the subgroup generated by  $s_1, \dots, s_{n-1}$  which is isomorphic to  $S_n$ . Since  $\eta_2(s_i) = t_i$  for  $i = 1, 2, \dots, n-1$ , it follows that  $\eta_2(\chi(x)) = 1$  if and only if  $\chi(x) = 1$  and then  $\eta_2$  is injective on  $\chi(H_n)$ . Therefore, if  $x$  belongs to  $H_n \cap VP_n$  then  $x$  belongs to  $\ker \chi$ .  $\square$

There is also another possible definition of  $EP_n$  as a generalisation of the classical pure braid group  $P_n$  (actually, it contains properly the normal closure of  $P_n$  in  $VB_n$ ).

**Proposition 24.** *Let  $\varepsilon : H_n \rightarrow S_n$  be the map defined by  $\varepsilon(x_{i,j}) = \varepsilon(x_{j,i}) = \rho_i^{\rho_{i+1} \cdots \rho_{j-1}}$  for  $1 \leq i < j \leq n$ . The morphism  $\varepsilon$  is well defined, the group  $EP_n$  is isomorphic to  $\ker \varepsilon$  and the normal closure of  $P_n$  in  $VB_n$  is properly included in  $EP_n$ .*

*Proof.* The morphism  $\varepsilon$  coincides with the restriction to  $H_n$  of the morphism  $\nu$  and therefore  $EP_n$  is isomorphic to  $\ker \varepsilon$ .

Denote by  $\langle\langle P_n \rangle\rangle_{VB_n}$  the normal closure of  $P_n$  in  $VB_n$ . Remark that  $\varepsilon(x_{1,2}^2) = 1$ . Since  $\langle\langle P_n \rangle\rangle_{VB_n}$  is actually the normal closure of  $\sigma_1^2 = x_{1,2}^2$ , one deduces that  $\langle\langle P_n \rangle\rangle_{VB_n} \subset EP_n$ .

On the other hand, let us consider the following exact sequence:

$$1 \rightarrow \langle\langle P_n \rangle\rangle_{VB_n} \rightarrow VB_n \rightarrow HB_n \rightarrow 1 ,$$

where  $HB_n$  is the group obtained adding relations  $\sigma_i^2 = 1$  to the group presentation of  $VB_n$ . The group  $HB_n$  contains elements of infinite order (for instance, consider the element  $(\sigma_i \rho_i)^2$ ). Therefore it is not isomorphic to  $M_n$  and we deduce that  $\langle\langle P_n \rangle\rangle_{VB_n}$  does not coincide with  $EP_n$ .  $\square$

## 8. FINITE TYPE INVARIANTS FOR VIRTUAL BRAIDS

We recall a possible definition of finite type invariants for classical braids, which is the algebraical version of the usual definition via singular braids (see [P] or Section 1.3 and Proposition 2.1 of [GP] in the case of surface braids).

In the following  $A$  will denote an abelian group. An invariant of braids is a set mapping  $v : B_n \rightarrow A$ . Any invariant  $v : B_n \rightarrow A$  extends by linearity to a morphism of  $\mathbb{Z}$ -modules  $v : \mathbb{Z}[B_n] \rightarrow A$ .

Now let  $V$  be the two-sided ideal of  $\mathbb{Z}[B_n]$  generated by  $\{\sigma_i - \sigma_i^{-1}, | i = 1, \dots, n-1\}$  and let  $V^d$  the  $d$ -th power of  $V$ . We obtain this way a filtration

$$\mathbb{Z}[B_n] \supset V \supset V^2 \supset \dots$$

that we call *Goussarov-Vassiliev filtration* for  $B_n$ .

A *finite type (Goussarov-Vassiliev) invariant of degree  $d$*  is a morphism of  $\mathbb{Z}$ -modules  $v : \mathbb{Z}[B_n] \rightarrow A$  which vanishes on  $V^{d+1}$ .

The Goussarov-Vassiliev filtration for  $B_n$  corresponds to the  $I$ -adic filtration of  $P_n$  (i.e., the filtration associated to the augmentation ideal of  $P_n$ ). More precisely, we denote by  $p$  the canonical projection of  $B_n$  on  $S_n$  and we recall that the set section  $s : S_n \rightarrow B_n$  sending  $\rho_i$  to  $\sigma_i$  (for  $i = 1, \dots, n-1$ ) determines an isomorphism of  $\mathbb{Z}$ -modules  $\Pi : \mathbb{Z}[B_n] \rightarrow \mathbb{Z}[P_n] \otimes \mathbb{Z}[S_n]$  defined as

$$\Pi(\beta) = \beta((p \circ s)(\beta))^{-1} \otimes p(\beta), \text{ for } \beta \in B_n$$

**Proposition 25.** (Papadima [P]). *The additive isomorphism  $\Pi : \mathbb{Z}[B_n] \rightarrow \mathbb{Z}[P_n] \otimes \mathbb{Z}[S_n]$  sends isomorphically  $V^d$  to  $I^d(P_n) \otimes \mathbb{Z}[S_n]$  for all  $d \in \mathbb{N}^*$ , where  $I^d(P_n)$  is the  $d$ -th power of the augmentation ideal of  $P_n$ .*

González-Meneses and Paris proved a similar proposition for braid groups on closed surfaces (Proposition 2.2 of [GP]).

In the case of virtual braids we can define a new notion of finite type invariant. This notion was introduced in [GPV] for virtual knots. Let  $J$  be the two-sided ideal of  $\mathbb{Z}[VB_n]$  generated by  $\{\sigma_i - \rho_i | i = 1, \dots, n-1\}$ .

We obtain this way a filtration

$$\mathbb{Z}[VB_n] \supset J \supset J^2 \supset \dots$$

that we call *Goussarov-Polyak-Viro filtration* for  $VB_n$ . We will call Goussarov-Polyak-Viro (GPV) invariant of degree  $d$  a morphism of  $\mathbb{Z}$ -modules  $v : \mathbb{Z}[VB_n] \rightarrow A$  which vanishes on  $J^{d+1}$ .

This notion of invariant corresponds to the remark that starting with a virtual knot and replacing finitely many crossings (positive or negative) we eventually get the (virtual) unknot.

On the other hand, one can also remark that a GPV invariant restricted to classical braids is a Goussarov-Vassiliev invariant. In fact, since  $(\sigma_i - \rho_i) - (\sigma_i^{-1} - \rho_i) = (\sigma_i - \sigma_i^{-1})$  one deduces that  $J^d \supset V^d$  where  $V^d$  is the  $d$ -th power of  $V$ , the two-sided ideal of  $\mathbb{Z}[VB_n]$  generated by  $\{\sigma_i - \sigma_i^{-1}, | i = 1, \dots, n-1\}$ .



The Goussarov-Polyak-Viro filtration for  $VB_n$  corresponds to the  $I$ -adic filtration of  $VP_n$ . The map  $\omega : VB_n \rightarrow VP_n \rtimes S_n$  defined in Proposition 11 determines an isomorphism of  $\mathbb{Z}$ -algebras  $\Omega : \mathbb{Z}[VB_n] \rightarrow \mathbb{Z}[VP_n] \otimes \mathbb{Z}[S_n]$ , where  $\mathbb{Z}[VP_n] \otimes \mathbb{Z}[S_n]$  carries the natural structure of  $\mathbb{Z}$ -algebra induced by the semi-direct product  $VP_n \rtimes S_n$ .

**Proposition 26.** *The  $\mathbb{Z}$ -algebras isomorphism  $\Omega : \mathbb{Z}[VB_n] \rightarrow \mathbb{Z}[VP_n] \otimes \mathbb{Z}[S_n]$  sends isomorphically  $J^d$  to  $I^d(VP_n) \otimes \mathbb{Z}[S_n]$  for all  $d \in \mathbb{N}^*$ .*

*Proof.* Let us denote  $VB_n$  by  $B$ . In order to prove the Proposition we need only to verify that:

$$J^d = BI^dB = BI^d = I^dB.$$

In fact using this equivalences we can repeat word by word the proof of Proposition 2.2 in [GP] and therefore prove the claim.

Now, since  $VP_n$  is a normal subgroup it suffices to prove that  $J = BIB$ . The inclusion  $J \subset BIB$  is obvious. In order to prove the other inclusion we only have to show that  $p - 1 \in J$  for  $p \in VP_n$ . Now let  $p_0 \in VP_n$ , let  $m$  denote the number of  $\sigma_i^{\pm 1}$  (for  $i = 1, \dots, n-1$ ) in the word  $p_0$  and let  $p_k$  the word obtained replacing each of the first  $k$  letters  $\sigma_i^{\pm 1}$  (for  $i = 1, \dots, n-1$ ) by  $\rho_i$  in  $p_0$ . The word  $p_m$  is the identity in  $VB_n$  and therefore  $p_0 - 1 = p_0 - \sum_{k=1}^{m-1} (p_k - p_{k-1}) - p_m = \sum_{l=0}^{m-1} (p_l - p_{l+1})$  belongs to  $J$ .  $\square$

## REFERENCES

- [AJ] M. A. Albar and D. L. Johnson, The centre of the circular braid group, *Math. Japonica*, **30**(1985), 641–645.
- [B] V. Bardakov, The virtual and universal braids, *Fund. Math.* **181**(2004), 1–18.
- [BE] L. Bartholdi, B. Enriquez, P. Etingof and E. Rains, Groups and Lie Algebras corresponding to the Yang Baxter equations, math.RA/0509661.
- [BF] P. Bellingeri and L. Funar, Braids on surfaces and finite type invariants, C. R. Math. Acad. Sci. Paris, **338**(2004), 157–162.
- [BGG] P. Bellingeri, J. Guaschi and S. Gervais, Lower central series for surface braids, preprint math.GT/0512155.
- [CCP] D. Cohen, F. Cohen and S. Prassidis, Centralizers of Lie algebras associated to the descending lower central series of certain poly-free groups, math.GT/0603470.
- [FR] M. Falk and R. Randell, The lower central series of a fiber type arrangement *Invent. Math.*, **82** (1985), 77–88.
- [FR2] M. Falk and R. Randell, Pure braid groups and products of free groups. *Contemp. Math.*, **78**(1988), 217–228.
- [F] R. H. Fox, Free differential calculus I: Derivation in the Free Group Ring, *Ann. of Math.*, **57**(1953), 547–560.
- [G] A. M. Gaglione, Factor groups of the lower central series for special free products, *J. Algebra*, **37**(1975), 172–185.
- [GL] S. Garoufalidis and J. Levine, Finite type 3-manifold invariants and the structure of the Torelli group I, *Invent. Math.*, **131**(1998), 541–594
- [GP] J. González-Meneses et L. Paris, Vassiliev Invariants for braids on surfaces, *Trans. A.M.S.*, **356** n° 1 (2004), 219–243.

- [GoL] E. A. Gorin and V. Ja. Lin, Algebraic equations with continuous coefficients and some problems of the algebraic theory of braids, *Math. USSR Sbornik*, **7**(1969), 569–596.
- [GPV] M. Goussarov, M. Polyak and O. Viro, Finite type invariants of classical and virtual knots, *Topology*, **39** n° 5 (2000), 1045–1068.
- [H] R. Hain, Infinitesimal presentations of the Torelli groups, *J. Amer. Math. Soc.*, **10** n° 3 (1997), 597–651.
- [K] S. Kamada, Invariants of virtual braids and a remark on left stabilisations and virtual exchange moves, *Kobe J. Math.*, **21**(2004), 33–49.
- [Ka] L. Kauffman, Virtual knot theory, *Eur. J. Comb.*, **20** n° 7 (1999), 663–690.
- [KMS] W. Magnus, A. Karrass, D. Solitar, Combinatorial group theory, Interscience Publishers, New York, 1996.
- [M] A. I. Malcev, Generalized nilpotent algebras and their associated groups, *Mat. Sbornik N.S.*, **25** (1949), 347–366.
- [Ma] I. Marin, On the residual nilpotence of pure Artin groups, *J. Group Theory*, to appear, math.GR/0502120.
- [MR] R. Mura and A. Rhemtulla, Orderable groups, Lecture Notes in Pure and Applied Mathematics **27**, Marcel Dekker, New York, 1977.
- [P] S. Papadima, The universal finite-type invariant for braids, with integer coefficients, *Topology Appl.*, **118** (2002), 169–185.
- [Pa] I. Bir S. Passi, Group rings and Their Augmentation Ideals, Lecture Notes In Mathematics **715**, Springer Verlag, Berlin Heidelberg New York, 1979.
- [R] L. Rabenda, mémoire de DEA, Université de Bourgogne (2003).
- [S] J. Stallings, Homology and Central Series of Groups, *J. Algebra*, **2**(1965), 170–181.
- [V] V. V. Vershinin, On homology of virtual braids and Burau representation, *J. Knot Theory Ramifications*, **10** n° 5 (2001), 795–812.

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